3.3 Decomposition of the Non-linear Term. Associated Equations

This section details the various existing decompositions of the non-linear term and the associated equations.

3.3.1 Leonard's Decomposition

Expression in Physical Space. Leonard [186] expresses the non-linear term in the form of a triple summation:

$$\overline{u_i u_j} = \overline{(\overline{u}_i + u'_i)(\overline{u}_j + u'_j)} \quad (3.13)$$

$$= \overline{\overline{u}_i\overline{u}_j} + \overline{\overline{u}_i u'_j} + \overline{\overline{u}_j u'_i} + \overline{u'_i u'_j} \quad . \quad (3.14)$$

The non-linear term is now written entirely as a function of the filtered quantity $\overline{\mathbf{u}}$ and the fluctuation $\mathbf{u}'$. We then have two versions of this [347].

The first considers that all the terms appearing in the evolution equations of a filtered quantity must themselves be filtered quantities, because the simulation solution has to be the same for all the terms. The filtered momentum equation is then expressed:

$$\frac{\partial \overline{u}_i}{\partial t} + \frac{\partial}{\partial x_j}\left(\overline{\overline{u}_i\overline{u}_j}\right) = -\frac{\partial \overline{p}}{\partial x_i} + \nu\frac{\partial}{\partial x_j}\left(\frac{\partial \overline{u}_i}{\partial x_j} + \frac{\partial \overline{u}_j}{\partial x_i}\right) - \frac{\partial \tau_{ij}}{\partial x_j} \quad , \quad (3.15)$$

in which the subgrid tensor τ, grouping together all the terms that are not exclusively dependent on the large scales, is defined as:

$$\tau_{ij} = C_{ij} + R_{ij} = \overline{u_i u_j} - \overline{\overline{u}_i\overline{u}_j} \quad , \quad (3.16)$$

where the cross-stress tensor, C, which represents the interactions between large and small scales, and the Reynolds subgrid tensor, R, which reflects the interactions between subgrid scales, are expressed:

$$C_{ij} = \overline{\overline{u}_i u'_j} + \overline{\overline{u}_j u'_i} \quad , \quad (3.17)$$

$$R_{ij} = \overline{u'_i u'_j} \quad . \quad (3.18)$$

In the following, this decomposition will be called double decomposition.

The other point of view consists of considering that it must be possible to evaluate the terms directly from the filtered variables. But the $\overline{\overline{u}_i\overline{u}_j}$ term cannot be calculated directly because it requires a second application of the filter. To remedy this, Leonard proposes a further decomposition:

$$\overline{\overline{u}_i\overline{u}_j} = \left(\overline{\overline{u}_i\overline{u}_j} - \overline{u}_i\overline{u}_j\right) + \overline{u}_i\overline{u}_j$$

$$= L_{ij} + \overline{u}_i\overline{u}_j \quad . \quad (3.19)$$

The new L term, called Leonard tensor, represents interactions among the large scales. Using this new decomposition, the filtered momentum equation becomes:

$$\frac{\partial \overline{u}_i}{\partial t} + \frac{\partial}{\partial x_j}\left(\overline{u}_i\overline{u}_j\right) = -\frac{\partial \overline{p}}{\partial x_i} + \nu\frac{\partial}{\partial x_j}\left(\frac{\partial \overline{u}_i}{\partial x_j} + \frac{\partial \overline{u}_j}{\partial x_i}\right) - \frac{\partial \tau_{ij}}{\partial x_j} \quad . \quad (3.20)$$

The subgrid tensor τ, which now groups all the terms that are not expressed directly from $\overline{u}$, takes the form:

$$\tau_{ij} = L_{ij} + C_{ij} + R_{ij} = \overline{u_i u_j} - \overline{u}_i\overline{u}_j \quad . \quad (3.21)$$

This decomposition will be designated hereafter the Leonard or triple decomposition. Equation (3.20) and the subgrid term τ_{ij} defined by (3.21) can be obtained directly from the Navier–Stokes equations without using the Leonard decomposition for this. It should be noted that the term $\overline{u}_i\overline{u}_j$ is a quadratic term and that it contains frequencies that are in theory higher than each of the terms composing. So in order to represent it completely, more degrees of freedom are needed than for each of the terms $\overline{u}_i$ and $\overline{u}_j$[4].

We may point out that, if the filter is a Reynolds operator, then the tensors C_{ij} and L_{ij} are identically zero[5] and the two decompositions are then equivalent, since the subgrid tensor is reduced to the tensor R_{ij}.

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4 In practice, if the large-eddy simulation filter is associated with a given computational grid on which the Navier–Stokes equations are resolved, this means that the grid used for composing the $\overline{u}_i\overline{u}_j$ product has to be twice as fine (in each direction of space) as the one used to represent the velocity field. If the product is composed on the same grid, then only the $\overline{\overline{u}_i\overline{u}_j}$ term can be calculated.

5 It is recalled that if the filter is a Reynolds operator, then we have the three following properties (see Appendix A):

$$\overline{\overline{\mathbf{u}}} = \overline{\mathbf{u}}, \ \overline{\mathbf{u}'} = 0, \ \overline{\overline{\mathbf{u}}\,\overline{\mathbf{u}}} = \overline{\mathbf{u}}\,\overline{\mathbf{u}} \quad ,$$

whence

$$C_{ij} = \overline{\overline{u}_i u'_j} + \overline{\overline{u}_j u'_i}$$
$$= \overline{\overline{u}}_i\overline{u'}_j + \overline{\overline{u}}_j\overline{u'}_i$$
$$= 0 \quad ,$$

$$L_{ij} = \overline{\overline{u}_i\overline{u}_j} - \overline{u}_i\overline{u}_j$$
$$= \overline{\overline{u}}_i\overline{\overline{u}}_j - \overline{u}_i\overline{u}_j$$
$$= 0 \quad .$$

Writing the Navier–Stokes equations (3.1) in the symbolic form

$$\frac{\partial \mathbf{u}}{\partial t} = \mathcal{NS}(\mathbf{u}) \quad , \tag{3.22}$$

the filtered Navier–Stokes equations are expressed

$$G\star\frac{\partial \mathbf{u}}{\partial t} = \frac{\partial \overline{\mathbf{u}}}{\partial t} = G\star\mathcal{NS}(\mathbf{u}) \tag{3.23}$$

$$= \mathcal{NS}(\overline{\mathbf{u}}) + [G\star, \mathcal{NS}](\mathbf{u}) \quad , \tag{3.24}$$

where $[.,.]$ is the commutator operator introduced in Sect. 2.1.2. We note that the subgrid tensor corresponds to the commutation error between the filter and the non-linear term. Introducing the bilinear form $B(\cdot,\cdot)$:

$$B(u_i, u_j) \equiv u_i u_j \quad , \tag{3.25}$$

we get

$$\tau_{ij} = [G\star, B](u_i, u_j) \quad . \tag{3.26}$$

Double decomposition (3.16) leads to the following equation for the resolved kinetic energy $q_r^2 = \overline{u}_i\overline{u}_i/2$:

$$\begin{aligned}\frac{\partial q_r^2}{\partial t} = & \underbrace{\overline{\overline{u}_i\overline{u}_j}\frac{\partial \overline{u}_i}{\partial x_j}}_{I} + \underbrace{\tau_{ij}\frac{\partial \overline{u}_i}{\partial x_j}}_{II} - \underbrace{\nu\frac{\partial \overline{u}_i}{\partial x_j}\frac{\partial \overline{u}_i}{\partial x_j}}_{III} \\ & - \underbrace{\frac{\partial}{\partial x_i}(\overline{u}_i\overline{p})}_{IV} + \underbrace{\frac{\partial}{\partial x_i}\left(\nu\frac{\partial q_r^2}{\partial x_i}\right)}_{V} \\ & - \underbrace{\frac{\partial}{\partial x_j}\left(\overline{u}_i\overline{\overline{u}_i\overline{u}_j}\right)}_{VI} - \underbrace{\frac{\partial}{\partial x_j}(\overline{u}_i\tau_{ij})}_{VII} \quad .\end{aligned} \tag{3.27}$$

This equation shows the existence of several mechanisms exchanging kinetic energy at the resolved scales:

- I - production
- II - subgrid dissipation
- III - dissipation by viscous effects
- IV - diffusion by pressure effect
- V - diffusion by viscous effects
- VI - diffusion by interaction among resolved scales
- VII - diffusion by interaction with subgrid modes.

Leonard's decomposition (3.21) can be used to obtain the similar form:

$$\begin{aligned}\frac{\partial q_r^2}{\partial t} = & -\underbrace{\frac{\partial q_r^2\overline{u}_j}{\partial x_j}}_{VIII} + \underbrace{\tau_{ij}\frac{\partial \overline{u}_i}{\partial x_j}}_{IX} - \underbrace{\nu\frac{\partial \overline{u}_i}{\partial x_j}\frac{\partial \overline{u}_i}{\partial x_j}}_{X} \\ & - \underbrace{\frac{\partial}{\partial x_i}(\overline{u}_i\overline{p})}_{XI} + \underbrace{\frac{\partial}{\partial x_i}\left(\nu\frac{\partial q_r^2}{\partial x_i}\right)}_{XII} \\ & + \underbrace{\overline{u}_i\overline{u}_j\frac{\partial \overline{u}_i}{\partial x_j}}_{XIII} - \underbrace{\frac{\partial}{\partial x_j}(\overline{u}_i\tau_{ij})}_{XIV} \quad .\end{aligned} \tag{3.28}$$

This equation differs from the previous one only in the first and sixth terms of the right-hand side, and in the definition of tensor τ:

- $VIII$ - advection
- IX - *idem II*
- X - *idem III*
- XI - *idem IV*
- XII - *idem V*
- $XIII$ - production
- XIV - *idem VII*

The momentum equation for the small scales is obtained by subtracting the large scale equation from the unfiltered momentum equation (3.1), making, for the double decomposition:

$$\begin{aligned}\frac{\partial u_i'}{\partial t} + \frac{\partial}{\partial x_j}\left((\overline{u}_i + u_i')(\overline{u}_j + u_j') - \overline{\overline{u}_i\overline{u}_j} - \tau_{ij}\right) = & -\frac{\partial p'}{\partial x_i} \\ & + \nu\frac{\partial}{\partial x_j}\left(\frac{\partial u_i'}{\partial x_j} + \frac{\partial u_j'}{\partial x_i}\right) \quad ,\end{aligned} \tag{3.29}$$

and, for the triple decomposition:

$$\begin{aligned}\frac{\partial u_i'}{\partial t} + \frac{\partial}{\partial x_j}\left((\overline{u}_i + u_i')(\overline{u}_j + u_j') - \overline{u}_i\overline{u}_j - \tau_{ij}\right) = & -\frac{\partial p'}{\partial x_i} \\ & + \nu\frac{\partial}{\partial x_j}\left(\frac{\partial u_i'}{\partial x_j} + \frac{\partial u_j'}{\partial x_i}\right) \quad .\end{aligned} \tag{3.30}$$

The subgrid kinetic energy $q_{sgs}^2 = \overline{u_k'u_k'}/2$ equation obtained by multiplying (3.30) by u_i' and filtering the relation thus derived is expressed:

$$
\begin{aligned}
\frac{\partial q_{\mathrm{sgs}}^2}{\partial t} &= \underbrace{-\frac{\partial}{\partial x_j}\left(q_{\mathrm{sgs}}^2\overline{u}_j\right)}_{XV} \underbrace{-\frac{1}{2}\frac{\partial}{\partial x_j}\left(\overline{u_iu_iu_j}-\overline{u}_j\overline{u_iu_i}\right)}_{XVI} \underbrace{-\frac{\partial}{\partial x_j}\left(\overline{pu_j}-\overline{p}\,\overline{u}_j\right)}_{XVII} \\
&+ \underbrace{\frac{\partial}{\partial x_j}\left(\nu\frac{\partial q_{\mathrm{sgs}}^2}{\partial x_j}\right)}_{XVIII} + \underbrace{\frac{\partial}{\partial x_j}\left(\tau_{ij}\overline{u}_i\right)}_{XIX} \\
&- \underbrace{\nu\left(\overline{\frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_j}}-\frac{\partial\overline{u}_i}{\partial x_j}\frac{\partial\overline{u}_i}{\partial x_j}\right)}_{XX} - \underbrace{\tau_{ij}\frac{\partial\overline{u}_i}{\partial x_j}}_{XXI}
\end{aligned}
\qquad (3.31)
$$

- XV - advection
- XVI - turbulent transport
- $XVII$ - diffusion by pressure effects
- $XVIII$ - diffusion by viscous effects
- XIX - diffusion by subgrid modes
- XX - dissipation by viscous effects
- XXI - subgrid dissipation.

For the double decomposition, equation (3.29) leads to:

$$
\begin{aligned}
\frac{\partial q_{\mathrm{sgs}}^2}{\partial t} &= \underbrace{-\frac{\partial}{\partial x_j}\left(\overline{u_iu_iu_j}-\overline{\overline{u}_i\overline{\overline{u}_i\overline{u}_j}}\right)}_{XXII} + \underbrace{\overline{u_iu_j\frac{\partial u_i}{\partial x_j}}-\overline{\overline{\overline{u}_i\overline{u}_j}\frac{\partial\overline{u}_i}{\partial x_j}}}_{XXIII} \\
&\underbrace{-\frac{\partial}{\partial x_j}\left(\overline{pu_j}-\overline{\overline{p}\,\overline{u}_j}\right)}_{XXIV} + \underbrace{\nu\left(\overline{u_i\frac{\partial^2u_i}{\partial x_j^2}}-\overline{\overline{u}_i\frac{\partial^2\overline{u}_i}{\partial x_j^2}}\right)}_{XXV} \\
&+ \underbrace{\frac{\partial}{\partial x_j}\left(\overline{\tau_{ij}\overline{u}_i}\right)-\overline{\tau_{ij}\frac{\partial\overline{u}_i}{\partial x_j}}}_{XXVI} \quad ,
\end{aligned}
\qquad (3.32)
$$

with:

- $XXII$ - turbulent transport
- $XXIII$ - production
- $XXIV$ - diffusion by pressure effects
- XXV - viscous effects
- $XXVI$ - subgrid dissipation and diffusion

It is recalled that, if the filter used is not positive, the generalized subgrid kinetic energy q_{gsgs}^2 defined as the half-trace of the subgrid tensor,

$$q_{\mathrm{gsgs}}^2 = \tau_{kk}/2 \quad ,$$

can admit negative values locally (see Sect. 3.3.5). If the filter is a Reynolds operator, the subgrid tensor is then reduced to the subgrid Reynolds tensor and the generalized subgrid kinetic energy is equal to the subgrid kinetic energy, *i.e.*

$$q_{\mathrm{sgs}}^2 \equiv \frac{1}{2}\overline{u_i'u_i'} = q_{\mathrm{gsgs}}^2 \equiv \tau_{kk}/2 \quad . \qquad (3.33)$$

Expression in Spectral Space. Both versions of the Leonard decomposition can be transcribed in the spectral space. Using the definition of the fluctuation $\widehat{\mathbf{u}}'(\mathbf{k})$ as

$$\widehat{u}_i'(\mathbf{k}) = (1-\widehat{G}(\mathbf{k}))\widehat{u}_i(\mathbf{k}) \quad , \qquad (3.34)$$

the filtered non-linear term $\widehat{G}(\mathbf{k})T_i(\mathbf{k})$ is expressed, for the triple decomposition:

$$
\begin{aligned}
\widehat{G}(\mathbf{k})T_i(\mathbf{k}) &= M_{ijm}(\mathbf{k})\int\int\widehat{G}(\mathbf{p})\widehat{G}(\mathbf{q})\widehat{u}_j(\mathbf{p})\widehat{u}_m(\mathbf{q})\delta(\mathbf{k}-\mathbf{p}-\mathbf{q})d^3\mathbf{p}d^3\mathbf{q} \\
&- M_{ijm}(\mathbf{k})\int\int(1-\widehat{G}(\mathbf{k}))\widehat{G}(\mathbf{p})\widehat{G}(\mathbf{q}) \\
&\qquad\times\widehat{u}_j(\mathbf{p})\widehat{u}_m(\mathbf{q})\delta(\mathbf{k}-\mathbf{p}-\mathbf{q})d^3\mathbf{p}d^3\mathbf{q} \\
&+ M_{ijm}(\mathbf{k})\int\int\widehat{G}(\mathbf{k})\left(\widehat{G}(\mathbf{p})(1-\widehat{G}(\mathbf{q}))+\widehat{G}(\mathbf{q})(1-\widehat{G}(\mathbf{p}))\right) \\
&\qquad\times\widehat{u}_j(\mathbf{p})\widehat{u}_m(\mathbf{q})\delta(\mathbf{k}-\mathbf{p}-\mathbf{q})d^3\mathbf{p}d^3\mathbf{q} \\
&+ M_{ijm}(\mathbf{k})\int\int\widehat{G}(\mathbf{k})\left((1-\widehat{G}(\mathbf{q}))(1-\widehat{G}(\mathbf{p}))\right) \\
&\qquad\times\widehat{u}_j(\mathbf{p})\widehat{u}_m(\mathbf{q})\delta(\mathbf{k}-\mathbf{p}-\mathbf{q})d^3\mathbf{p}d^3\mathbf{q} \quad .
\end{aligned}
\qquad (3.35)
$$

The first term of the right-hand side corresponds to the contribution $\overline{u}_i\overline{u}_j$, the second to the Leonard tensor L, the third to the cross stresses represented by the tensor C, and the fourth to the subgrid Reynolds tensor R. This is illustrated by Fig. 3.1.

The double decomposition is derived by combination of the first two terms of the right-hand side of (3.35):

$$
\begin{aligned}
\widehat{G}(\mathbf{k})T_i(\mathbf{k}) &= M_{ijm}(\mathbf{k})\int\int\widehat{G}(\mathbf{p})\widehat{G}(\mathbf{q})\widehat{G}(\mathbf{k}) \\
&\qquad\times\widehat{u}_j(\mathbf{p})\widehat{u}_m(\mathbf{q})\delta(\mathbf{k}-\mathbf{p}-\mathbf{q})d^3\mathbf{p}d^3\mathbf{q} \\
&+ M_{ijm}(\mathbf{k})\int\int\widehat{G}(\mathbf{k})\left(\widehat{G}(\mathbf{p})(1-\widehat{G}(\mathbf{q}))+\widehat{G}(\mathbf{q})(1-\widehat{G}(\mathbf{p}))\right) \\
&\qquad\times\widehat{u}_j(\mathbf{p})\widehat{u}_m(\mathbf{q})\delta(\mathbf{k}-\mathbf{p}-\mathbf{q})d^3\mathbf{p}d^3\mathbf{q} \\
&+ M_{ijm}(\mathbf{k})\int\int\widehat{G}(\mathbf{k})\left((1-\widehat{G}(\mathbf{q}))(1-\widehat{G}(\mathbf{p}))\right) \\
&\qquad(\mathbf{p})\widehat{u}_m(\mathbf{q})\delta(\mathbf{k}-\mathbf{p}-\mathbf{q})d^3\mathbf{p}d^3\mathbf{q} \quad .
\end{aligned}
\qquad (3.36)
$$